Critical exponents for structural phase transitions in a complex plasma

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Abstract. The critical instability of two dust particles levitating in the complex plasma sheath of a radio-frequency discharge is considered. It is shown that the two-particle system has a critical point where the alignment symmetry is continuously broken as the system parameter is varied. The associated critical exponents are derived and found to belong to the Ising universality class. Another universality class is suggested for symmetry breaking of the radial and vertical confinement potentials.

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INTRODUCTION

The theory of critical phenomena has mostly been explored from the perspective of the statistical thermodynamics. In the so-called extensive system regime, where the number of interacting particles is of the order of Avogadro's number, the assumption of an infinite uniform system is justified. In non-extensive systems where the number of particles is fewer than $\sim 10^3$, the thermodynamic limit can no longer be assumed, since the extent of the interparticle interaction is comparable the size of the system. Complex plasma provides an ideal medium for studying phase transitions in non-extensive systems, with as little as two particles displaying extremely rich physics such as spontaneous symmetry breaking and universality.

RESULTS

The Order Parameter Exponent β

Below a critical radial confinement $\omega_{\rho,\mathrm{c}}$. The Hamiltonian has a stable equilibrium with the dust particles horizontally aligned ($\Delta z = 0$). Taylor expanding the Hamiltonian about the equilibrium position in the vertical interparticle separation order parameter Δz we obtain

$$\begin{aligned}\mathscr{H} &= \mathscr{H}'(0)\Delta z + \frac{1}{2}\mathscr{H}''(0)\Delta z^2 + \frac{1}{6}\mathscr{H}^{(3)}(0)\Delta z^3 + \frac{1}{24}\mathscr{H}^{(4)}(0)\Delta z^4 \\ &= \frac{1}{4}M(\omega_z^2 - \Omega^2\omega_\rho^2)\Delta z^2 + \frac{1}{24}\mathscr{H}^{(4)}(0)\Delta z^4 \end{aligned} \tag{1}$$

where Ω denotes the critical vertical to radial frequency ratio and the coefficient $\mathscr{H}^{(4)}(0)$ depends only on the internal parameters of the dust system. Treating ω_z as a constant, as required by constant sheath field, we may recast (1) as

$$\mathscr{H} = \frac{1}{4}M\omega_z^2(1 - \omega_\rho^2/\omega_{\rho,\mathrm{c}}^2)\Delta z^2 + \frac{1}{24}\mathscr{H}^{(4)}(0)\Delta z^4 \tag{2}$$

where $\omega_{\rho,\mathrm{c}}$ denotes the critical radial angular frequency for a given vertical angular frequency ω_z. Near the critical point, (2) becomes

$$\mathscr{H} = -\frac{1}{2}M\omega_z^2\zeta\Delta z^2 + \frac{1}{4}a_4\Delta z^4 \tag{3}$$

where we have written $a_4 \equiv \frac{1}{6}\mathscr{H}^{(4)}(0)$. Equation (3) corresponds to the Ginzburg-Landau Hamiltonian for the Ising model in the mean field approximation with zero external magnetic field strength $H = 0$ [2]. The equilibria of (3) are found using the minimum condition $\mathrm{d}\mathscr{H}/\mathrm{d}\Delta z = 0$,

$$0 = -M\omega_z^2\zeta\Delta z + a_4\Delta z^3. \tag{4}$$

For $\zeta < 0$, the only real solution is $\Delta z = 0$. That is, below the critical frequency, the dust particles remain aligned in the horizontal plane. Above the critical frequency $\zeta \geq 0$, the ground-state of the system bifurcates into two degenerate equilibria which are related by the $\Delta z \rightarrow -\Delta z$ symmetry of the Hamiltonian

$$\Delta z = \pm\sqrt{\frac{M\omega_z^2\zeta}{a_4}}. \tag{5}$$

The order parameter changes continuously as the frequency passes the critical point, with the critical exponent $\beta = 1/2$ defined by $\Delta z \propto |\zeta|^\beta$ characteristic of a continuous, or second-order phase transition. The Ising model of ferromagnetics undergoes such a transition from paramagnetic to ferromagnetic phase as $T \rightarrow T_c^+$ at zero external magnetic field strength. In this case, with the zero-field magnetisation $M|_{H=0} \propto |\varepsilon|^{1/2}$ as the order parameter. At non-zero magnetic field strength, the system loses its spin-reversal symmetry, so that the continuous phase transition can no longer occur. The external magnetic field H is said to be conjugated to the order parameter M.

The Response Exponent δ

Motivated by the external magnetic field of the Ising model, we search for a field conjugated to the order parameter Δz of the dust system. The asymmetric wake provides this field. Introducing an asymmetry between the wake charges of two particles ΔQ_w induces explicit symmetry breaking terms in the Hamiltonian

$$\mathscr{H} = a_1\Delta Q_\mathrm{w}\Delta z - \frac{1}{2}M\omega_z^2\zeta\Delta z^2 + \frac{1}{3}a_3\Delta Q_\mathrm{w}\Delta z^3 + \frac{1}{4}a_4\Delta z^4. \tag{6}$$

Note that it is necessary to replace each instance of Q_w in the expressions for the coefficients a_1 and a_4 by the sum of both wake charges, which is assumed to remain constant. The symmetry breaking terms skew the Hamiltonian so that the oblique equilibria lose their degeneracy and one becomes energetically favoured.

Along the critical isofrequency $\zeta = 0$, the equilibrium condition gives

$$\Delta Q_\mathrm{w} = -\frac{a_4\Delta z^3}{a_1 + a_3\Delta z^2}. \tag{7}$$

The first term in the Taylor expansion of (7) about $\Delta z = 0$ occurs at third order in Δz, and all subsequent terms occur at fifth order or higher. This provides the critical exponent $\delta = 3$ for the scaling relation $\Delta z \propto \Delta Q_\mathrm{w}^{1/\delta}$. If the wake field ΔQ_w passes zero at or above the critical frequency, then the dust system will change continously from one oblique equilibrium to the other. If this occurs below the critical frequency, the order parameter Δz develops a singularity at $\Delta Q_\mathrm{w} = 0$, resulting in a discontinuous jump in the order parameter, known as a first order phase transition.

The Susceptibility Exponent γ

The susceptibility χ of a system is defined as the linear response of the order parameter to infinitesimal changes in the conjugate field. The critical exponent γ is defined such that $\chi = (\partial M/\partial H)_{H\rightarrow 0} \propto |\varepsilon|^{-\gamma}$. In the mean-field approximation, the predicted value is $\gamma = 1$

The equilibria of the general Hamiltonian of the dust system (6) are analytically soluble. Differentiating the solution with respect to the wake difference ΔQ_w and taking the limit of the derivative as $\Delta Q_\mathrm{w} \rightarrow 0$ we obtain the critical exponent $\gamma = 1$ for the susceptibility $\chi = (\partial\Delta z/\partial\Delta Q_\mathrm{w})_{\Delta Q_\mathrm{w}\rightarrow 0} \propto |\zeta|^{-\gamma}$.

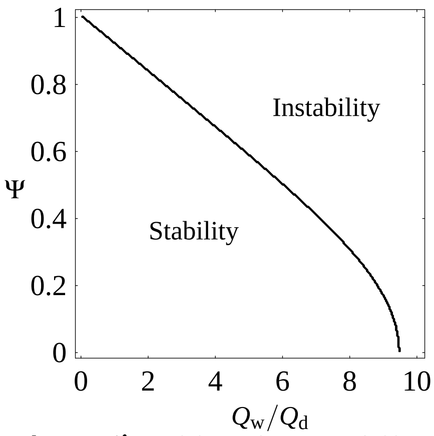

FIGURE 1. The numerical phase diagram ($\lambda_{\mathrm{D}} = 6.07 \times 10{-}4$, $M = 3.60 \times 10{-}14$, $Q_{\mathrm{d}} = 3.6 \times 103e$, $\ell = 1.5\lambda_{\mathrm{D}}$)

The Specific Heat Exponent α

This exponent describes the divergence of the specific heat $C \propto |\varepsilon|^{-\alpha}$ at the critical temperature. The value $\alpha = 0$ follows trivially from the definition of the specific heat of the dust system,

$$C = \frac{\partial \mathscr{H}}{\partial \zeta} \propto |\zeta|^{-0}. \tag{8}$$

CONFINEMENT SYMMETRY BREAKING

We now consider symmetry breaking of a different kind, namely breaking of confinement symmetry in the radial and vertical directions. The vertical ion flow in the sheath naturally provides a prevalent direction, reducing the symmetry of the system. In order to describe the extent of this symmetry breaking, we use the inverse critical frequency ratio $\Psi \equiv \Omega^{-1}$ as an order parameter. Linear perturbation theory suggests the following expression for the critical ratio

$$\Omega^2 = 1 + \frac{Q_{\mathrm{w}} Q_{\mathrm{d}}}{2\pi\varepsilon_0 M \omega_\rho^2} \left[\frac{\ell^2 \kappa}{(\ell^2 + \Delta x_0^2)^2} - \left(1 + \kappa\sqrt{\ell^2 + \Delta x_0^2}\right) \left(\frac{3\ell^2}{(\ell^2 + \Delta x_0^2)^{5/2}} + \frac{\ell^2 \kappa}{(\ell^2 + \Delta x_0^2)^2} \right) \right] \mathrm{e}^{-\kappa\sqrt{\ell^2 + \Delta x_0^2}}. \tag{9}$$

This is a generalisation of a previous result [1] to include the effects of linear electron screening. Rearranging (9) to eliminate ω_ρ explicitly from the right-hand side we obtain

$$\Psi^2 = 1 - \frac{Q_{\mathrm{w}} Q_{\mathrm{d}}}{2\pi\varepsilon_0 M \omega_z^2} \left[\frac{\ell^2 \kappa}{(\ell^2 + \Delta x_0^2)^2} - \left(1 + \kappa\sqrt{\ell^2 + \Delta x_0^2}\right) \left(\frac{3\ell^2}{(\ell^2 + \Delta x_0^2)^{5/2}} + \frac{\ell^2 \kappa}{(\ell^2 + \Delta x_0^2)^2} \right) \right] \mathrm{e}^{-\kappa\sqrt{\ell^2 + \Delta x_0^2}}. \tag{10}$$

Plotting Ψ as a function of Q_{w} at constant ω_z reveals a curve which terminates at a critical wake charge $Q_{\mathrm{w,c}}$. Given that the coordinates of the critical point in the (Q_{w}, Ψ) space are $(Q_{\mathrm{w,c}}, 0)$, it quickly follows that near the critical point

$$\Psi = \pm\sqrt{\frac{Q_{\mathrm{w,c}} - Q_{\mathrm{w}}}{Q_{\mathrm{w,c}}}} \tag{11}$$

and thus we obtain the critical exponent $\beta = 1/2$ defined by $\Psi \sim (-\theta)^\beta$, where $\theta = (Q_{\mathrm{w}} - Q_{\mathrm{w,c}})/Q_{\mathrm{w,c}}$. If we tentatively define the susceptibility

$$\chi_{\mathrm{spat}} = \left(\frac{\partial \omega_\rho}{\partial \omega_z} \right)_{\omega_z \to 0} \tag{12}$$

then, multiplying (10) by ω_z, partially differentiating with respect to ω_z and taking the limit as $\omega_z \to 0$ we obtain the critical exponent $\gamma = 1/2$ where $\chi_{\mathrm{spat}} \sim (-\theta)^{-\gamma}$.

To assist in the determination of the response exponent δ, we introduce the normalized resonant frequencies $\widetilde{\omega}_\rho = \omega_\rho/\omega_{\mathrm{z,c}}$ and $\widetilde{\omega}_{\mathrm{z}} = \omega_{\mathrm{z}}/\omega_{\mathrm{z,c}}$. In this notation, (10) becomes

$$\Psi = \sqrt{1 - \frac{\theta+1}{\widetilde{\omega}_z^2}}. \tag{13}$$

As the critical point is approached, we may write

$$\widetilde{\omega}_z = \frac{1}{\sqrt{1-\widetilde{\omega}_\rho^2}} \approx 1 + \frac{1}{2}\widetilde{\omega}_\rho^2$$

and therefore $\delta = 2$.

The critical exponents $\beta = 1/2$, $\gamma = 1/2$ and $\delta = 2$ satisfy the Widom equality

$$\gamma = \beta(\delta - 1)$$

suggestive of universality. Assuming that the Griffith's equality

$$\alpha + \beta(\delta + 1) = 2$$

holds true, we may assign the critical exponent $\alpha = 1/2$ for the specific heat capacity $C = \partial Q/\partial\theta \propto |\theta|^{-\alpha}$.

DISCUSSION AND CONCLUSION

The critical exponents for the horizontal alignment instability are the same as those of the mean field theory for thermodynamic systems such as the Ising model and Van der Waals theory. The dust system exhibits both a continuous (second order) phase transition at the critical resonant frequency, as well as discontinuous phase transitions induced by asymmetric wake fields. The exponents are independent of the plasma parameters such as the Debye length. Although we made use of the a the point-charge approximation to model the ion wake distribution, the results should apply equally well for any wake distribution since they depend only on the local approximation to wake potential up to third order.

The second order phase transition is easily observed for two identical particles in a uniform discharge. In order to observe the discontinuous first order phase transition between the oblique equilibria, consideration must be given to the experimental realisation of the wake charge asymmetry ΔQ_{w}. The wake asymmetry may come about by virtue of the non-uniformity of the discharge, however, dynamical wake charging in the plasma is necessary to observe the jump.

In terms of spatial symmetry breaking of the confinement fields, the critical frequency ratio Ψ for two identical dust particles provides an order parameter to describe the extent of the symmetry breaking by the dust-induced wake fields. The remarkable simplicity of the order parameter scaling $\Psi \sim (-\theta)^{1/2}$ near the critical wake charge tempts us to define other critical exponents for the susceptibility and the response. The fact that the exponents β, γ and δ satisfy the Widom equality gives strong support to the notion of universality. It is not clear, however, if the predicted exponent of $\alpha = 1/2$ can be derived from the Hamiltonian.

The role of fluctuations has not been considered. Thermal fluctuations in thermodynamic systems are responsible for changing the critical exponents from their mean field values. Fluctuations of grain charge in the dust system may allow us to define a correlation function, giving deeper insight into the universality class of this system.

The fact that the horizontal alignment instability belongs to the Ising universality class for thermodynamic systems suggests a common symmetry underlying these seemingly disparate systems. The new universality class for breaking of the external confinement symmetry needs to be explored in greater detail so that it can be compared to other systems which share the same symmetry.

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