On Particular Feature Of The Modulation Instability Of The Finite Amplitude Wave Near Threshold

Kuklin V.M.

Institute of High Technology, Kharkiv Karasin's National University, Svobody sq. 4, Kharkiv 61077,Ukraine

1. Introduction

It is well known that the finite amplitude waves in medium with cubic nonlinearity are unstable with excitation of the lateral spectra of perturbation [1]. Development of such instability [2-4] (introduced in [3]) at least on its early growth leads to modulation of the wave amplitude (see also [5]). This modulation instability of the finite amplitude waves on the surface of ocean has been examined thoroughly in the papers [6-9]. To observe the modulation instability in absorbing media (or under radiation loss conditions) an external source of energy is required, which has to support the finite amplitude waves.

In this paper the three basic characteristics of the modulation instability of the wave (pump wave) supported by external source in absorbing media are discussed.

First of all, the interaction between the modes of the perturbation spectrum is negligible [10-11]. Only an integral action on the pump wave by the perturbation spectrum is substantial. Therefore a linear stage of the instability transforms to the so-called quasilinear stage. At that stage of instability process the integrated intensity of perturbation spectrum independently of its width reaches (but not exceeds) some threshold level. From this moment the rate of change of their amplitudes (but not phases) becomes abruptly slower [12].

Deceleration is a result of depumping. Here depumping is a decrease of the pump wave amplitude. The depumping occurs due to an integral action of the perturbation spectrum on main mode. Hence the life time of the instability process is hundred or thousand times greater then a value of the inverse increment of linear theory. Under existing conditions the phases of perturbation spectrum modes locked by pump wave are able to form forced interference splashes, induced by pump wave. Incidentally, the noise level reduction [13] leads to a subsidiary deceleration of quasilinear stage of the modulation instability in media with and without wave motion [12,14].

The behavior of the separate mode amplitude of perturbation spectrum $a_n = a(k_n)$ near the instability threshold can be described by the reductive equation

$$da_n / dt = [\gamma_L(k_n) - \delta] \cdot a_n - 2\sum_m | a_m |^2 a_n , \ (m \neq 0) \qquad (1)$$

and at the same time $\gamma_L(k_n) \le \gamma_{LMAX}(a_0)$. a_0 – amplitude of main wave.

The quasilinear stage of instability starts with the achievement of the threshold intensity

$$(2\sum_{m\neq 0} | a_m |^2)_{THR} = a_0^{-2} D \propto [\gamma_{LMAX} - \delta] \qquad (2)$$

independently of the spectrum width.
is a nonlinear increment, D is a level of an imperfection, which is a small parameter as well. At the quasilinear stage strong inequality $|\gamma_{NL}| << \gamma_L - \delta$ determines anomalous retardation of the instability process. A slow change of the amplitudes of the unstable modes coupled up relatively fast behavior of the phases. The phase motion is able to form the interference splash or fine structure of the perturbation, induced by pump. This effect of the induced interference at the quasilinear stage of instability appears from the forced-phase locking by pump. Average amplitude of *N*-modes of the perturbation spectrum is $\overline{a_n} \propto a_0 \sqrt{D} / \sqrt{N}$ and the wave amplitude in the area of the interference splash is able to reach a value $a_0 \sqrt{D \cdot N}$. In case of $D << 1$ and $N >> 1$ the amplitude of the modulated wave in the area of the interference splash is a few times greater than such amplitude in ambient space [15,16].

In the second place, on the quasilinear stage of instability the pumping intensity is slowly decreasing. Here by the pumping intensity we mean the intensity of pump wave. The peripheral parts of the perturbation spectrum are putting down. Though some modes of the central parts of that spectrum keep slowly growing. Finally that process results in abnormal bandwidth narrowing of the instability and formation of the line spectrum of the modulated wave [10,11].

In most cases near threshold of instability the quasilinear operation is realized and coupled with effect of induced interference. It's obvious that this mechanism is responsible for formation of the fine structure of the laser pulses [15,16].

Thirdly, formation of the line spectrum of the modulated wave makes easier to fulfill the conditions for the next secondary modulation instability [17], which evolves in the same way. An increment of the modulation instability of nonlinear wave with the line spectrum is the largest [18,19]. The wave, which was modulated as a result of primary modulation instability, is exposed modulation by the secondary modulation instability on a large spatial and temporal scale. It is easy to see that each mature modulation keeps the shape of the initial pump wave on its own scale and in that way forms the self-similar structure [17] with the similarity coefficient $1/(1-\delta)$. The mature line spectrum in the wave-vector space will represent a fractal, when the cascade of all modulation instabilities will be complete. The spatial self-similar structures in the unstable media near the threshold of the modulation instability cause nonlinearity, nonequilibrium, energy absorption and have weak dependence on dispersion.

Below the development of a modulation instability cascade of the wave supported by external source in absorbing media with cubic nonlinearity are discussed.

2. Primary modulation instability

Let us assume that Lighhill equation [1] is correctly for slowly variable complex amplitude of the wave perturbation and it describes a nonlinear wave propagation

$$\frac{\partial A}{\partial t} = -\delta A - i\frac{\partial^2 A}{\partial x^2} - iA\,|\,A\,|^2 + g \qquad (3)$$

where δ is a decrement of oscillation damping, g is an external source, which supports the finite amplitude monochromatic wave A with the wavenumber $k = k_0$. The variables t, x are the normalized time and coordinate, correspondingly.

This equation and its generalization (Ginzburg –Landau equation , Newell equation [20]) describe the great number physical phenomena in the context well-defined restriction. Among them there are an evolution of a wind-induced wave, the wave dynamics of concentration during the conversion, the numerous instabilities in plasma, instability of the finite amplitude wave in medium with cubic nonlinearity and a whole series of other phenomena.

Let's use the unidimensional analysis. The equation (3) is presented in the form

$$\frac{\partial A_k}{\partial t} = -\delta A_k + ik^2 A_k - i\int dk_1 dk_2 dk_3 A_{k_1} A_{k_2} A_{k_3} \delta(k - k_1 - k_2 + k_3) + g \qquad (4)$$

where A_{k_i} is a Fourier form of perturbation amplitude A. Let a main mode be $u_0 \exp\{i\varphi_0 - ik_0 x\}$, where $u_0 = A_{k_0}, \varphi_0 = \varphi_{k_0}$ is an amplitude and a phase of the wave. The main mode is a pump wave for the spectrum of the instable modes. The spectrum of oscillation $u_n \exp\{i\varphi_n - ik_n x\}$ is excited as a result of instability. This oscillation is connected with the main mode by the spatial synchronism conditions $2k_0 = k_n + k_{-n}$, где $k_n = k_0 + K_n$, $k_{-n} = k_0 - K_n$ ($K_n > 0, K_n << k_0$).

Let us note that the wave numbers of the instable modes are symmetrically distributed relatively to a wave number of the main mode. The amplitude of the main mode in these conditions is determined from the equation

$$u_0 = -g\{-\delta - 2\sum_{m>0}^{N} u_m u_{-m} Sin\,\Phi_m\}^{-1} = 1/\{1 + \frac{2}{\delta}\sum_{m>0}^{N} u_m{}^2 Sin\,\Phi_m\} \qquad (5)$$

The summation here and below is carried out only with the positive indexes $m, n = 1,2,...N$, and $\Phi_n = 2\varphi_0 - \varphi_n - \varphi_{-n}$ is a phase of n-th channel of instability. Here Φ_0 is not in existence and $\Phi_n = \Phi_{-n}$, $u_n = u_{-n}$ [10]. Let us assume for simplicity $\delta = g$. The requirement of closeness to threshold of instability leads to the small parameter

$$D = \frac{2}{u_0^2}\sum_{m>0}^{N} u_m^2 \ , \qquad (6)$$

which at the same time define an imperfection D of the growing spatial structure [12]. The initial value of the phase of the main wave (n = 0) equals to zero. This phase is described by following equation

$$\frac{d\varphi_0}{dt} = +k_0{}^2 - u_0{}^2 - 4\sum_{m>0}^{N} u_m{}^2 - 2\sum_{m>0}^{N} u_m{}^2 Cos\Phi_m \ . \qquad (7)$$

A change of the amplitudes of the growing modes is defined by equations

$$\frac{du_n}{dt} = u_n\{-\delta + u_0^2 Sin\,\Phi_n\} \ . \qquad (8)$$

It is obvious, that reversal of the sign of n doesn't change the equations. The phases of the modes depend on sign of n.

$$\frac{d\varphi_n}{dt} = k_n^2 - 2(u_0^2 + 2\sum_{m>0}^{N} u_m^2 - \frac{1}{2}u_n^2) - u_0^2 Cos\Phi_n . \qquad (9)$$

For calculation it is necessary to know, how the n-th channel phase of instability is changing.

$$\frac{d\Phi_n}{dt} = \Delta_n + 2(u_0^2 - u_n^2) + 2(u_0^2 Cos\Phi_n - 2\sum_{m>0}^{N} u_m{}^2 Cos\Phi_m), \qquad (10)$$

where $\Delta_n = 2k_0^2 - k_n^2 - k_{-n}^2$. It is easy to see, that $\Delta_n = -2K_n{}^2$.

Linear theory. First of all let us note that for realization of the instability it is necessary at least that the phase of each n-th channel of instability Φ_n fast possesses the defined value $\Phi^*{}_n$.On the linear stage of the process $\Phi_n \approx \Phi_n^*$. The phase of n-th channel of instabilitypractically doesn't change and exponential growth of amplitude u_n is beginning. If the small values are neglected, one may find from (10) the value $Cos\Phi_n = Cos\Phi_n^* = (\Delta_n + 2u_0^2)/2u_0^2$, and define $Sin\Phi_n^* = (-\Delta_n^2 - 4\Delta_n u_0^2)^{1/2}/2u_0^2$. Therefore, the equation (8) in the linear approximation against the amplitudes of growing modes takes the form

$$\frac{du_n}{dt} = u_n\{-\delta + (-\Delta_n^2 - 4\Delta_n u_0^2)^{1/2}/2\} , \qquad (11)$$

and a linear increment of instability is equal to $\mathrm{Im}\,\omega = -\delta + (-\Delta_n^2 - 4\Delta_n u_0^2)^{1/2}/2$.

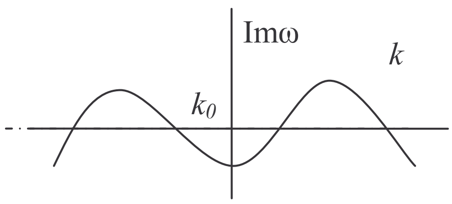

Fig.1. Increment of modulation instability Imω as a function of wavenumber *k* for $\delta \neq 0$

If $\Delta_n = -2u_0^2$, then increment reaches a maximum value which equals to $(1-\delta)$, where $0 < \delta < 1$. The interval of instability in the wave-vector space is determined by the requirement $\mathrm{Im}\,\omega > 0$ and is specified by the following inequality $-2(1+\sqrt{1-\delta^2}) < \Delta_n < -2(1-\sqrt{1-\delta^2})$. The set of equations (5), (8), (10) describes the modulation instability in case of the small exceeding of the instability threshold (i.e. if $1-\delta << 1$). At the same time the phases are located in a neighbourhood of $\Phi^*{}_n$, which are slowly changing when the perturbation amplitudes grow up and the pump level is reduced. The equation (7) and (9) allow to receive information about the behavior of phase of separate interactive modes.

Modulation of the main wave. For clarification of a character of growing spatial modulation of the main wave let examine an approximation theory, when the changes of phases $\Phi_n = \Phi^*{}_n$ are neglected. In that case one may use a small parameter (6) in order to receive the following expression for $Cos\Phi_n, Sin\Phi_n$

$$Cos\Phi_n = -\frac{\Delta_n + 2(u_0^2 - u_n^2)}{2u_0^2} , \quad Sin\Phi_n = (-\Delta_n^2 - 4u_0^2\Delta_n)^{1/2}/2u_0^2 \qquad (12)$$

The items proportional to $\frac{(\Delta_n + 2u_0^2)}{u_0^4}\sum_{m>0}^{N} u_m^2$ are neglected, since $\frac{(\Delta_n + 2u_0^2)}{2u_0^2} << 1$. This expression of the trigonometrical functions allow to find the equations for an amplitude

$$u_0 \approx \{1 - \frac{2}{\delta}\sum_{m>0}^{N} u_m{}^2\} , \qquad (13)$$

and a phase main mode

$$\frac{d\varphi_0}{dt} \approx k_0{}^2 - u_0{}^2 - 4\sum_{m>0}^{N} u_m{}^2 , \qquad (14)$$

For amplitude and phase of the growing modes the following equations are valid

$$\frac{du_n}{dt} = u_n\{-\delta + \frac{1}{2}(-\Delta_n^2 - 4\Delta_n)^{1/2} - \frac{4}{\delta}\sum_{m>0}^{N} u_m^2\}\ , \tag{15}$$

$$\frac{d\varphi_n}{dt} = k_n^2 - u_0^2 - 4\sum_{m>0}^{N} u_m^2 + \frac{\Delta_n}{2} = k_0^2 - u_0^2 - 4\sum_{m>0}^{N} u_m^2 + 2\frac{n}{|n|}k_0\sqrt{\frac{|\Delta_n|}{2}}\ , \tag{16}$$

The expression for modulated wave in the conditions of developed instability in thatapproximation is represented in the form

$$\begin{aligned} E(x,t) &= u_0 \exp\{-ik_0x + i\varphi_0(t)\} + \sum_{m>0}^{N}[u_m \exp\{-ik_mx + i\varphi_m(t)\} + u_{-m}\exp\{-ik_{-m}x + i\varphi_{-m}(t)\}] = \\ &= \exp\{-ik_0x + i\varphi_0(t)\}[u_0 + \sum_{m>0}^{N} u_m \cdot \exp\{\Phi_m/2)\} \cdot Cos\{K_m\xi - (\varphi_{m0} - \varphi_{-m0})/2\}] \end{aligned} \tag{17}$$

where $\xi = x - 2k_0 t$, $\varphi_{m0} = \varphi_m(t=0), \varphi_{-m0} = \varphi_{-m}(t=0)$ - initial phases of the modes.

Thus a second item in (17), i.e. a modulation of the main wave, represents a sum of periodic perturbation with a wave-length equal to $2\pi / K_n = 2\pi\sqrt{\frac{2}{|\Delta_n|}}$, which is in $k_0 / K_n = \sqrt{\frac{2k_0^2}{|\Delta_n|}}$ times greater then a length of the main mode. It is important that in,. this approximation all perturbations don't shift one relatively another.

3. Induced interference of the instability spectrum modes and formation of amplitude splashes

The slow change of channel phase (i.e. $d\Phi_n / dt \neq 0$) account for slow relative motion of perturbation with different wave-length (see the second item of (17)). Let us examine the process of splash formation of modulated wave amplitude. It follows from equation (15) that exponential growth of the amplitudes of instability spectrum stops when a second item of equation is approaching zero. A1so

$$2\sum_{m>0}^{N} u_m^2 \approx D \tag{18}$$

where $D(\delta)$ is some value, less than unit. At that moment the modulated wave (17) is formed. Modulated wave is composed of the main mode and a set of long-wave perturbations, which slowly shift one relatively another. A rate of instability evolution becomes slower sharply and the modes in the outlying parts of spectrum, which are long-wave and short-wave parts of spectrum, decrease their own amplitudes. The modes from a band center are slowly increasing. The instability spectrum gradually converges.

It is important to note that at a quasilinear stage of instability with decreasing number of modes N or spectrum width ΔK the equation (18) practically doesn't change. An average value of the mode amplitudes of instability spectrum rises. In case of discrete spectrum the expressions $2N \cdot (u^2)_{av} \approx D$ and $\bar{u} = \sqrt{(u^2)_{av}} \propto \sqrt{\frac{D}{2N}}$ are valid. In certain spatial domain will be formed a splash of modulated wave with an amplitude $N \cdot \bar{u} \propto \sqrt{N \cdot D/2}$. Is it possible to estimate a time of splash formation τ as

$$\tau \propto 4\pi / [(d\Phi/dt)_{\max} - (d\Phi/dt)_{\min}]. \tag{19}$$

To imagine the pattern of splash, one may use an approximation at the beginning of quasilinear stage of instability

$$2\sum_{m=1}^{N} u_m^2 = \frac{1}{\delta K}\int dK \cdot u_m^2 = \frac{N}{\Delta K}\int dK \cdot \frac{D}{N}\exp\{-2\frac{(K - K_{N/2})}{\Delta K}\}\ , \tag{20}$$

where $K_{N/2}$ is a central mode of modulation spectrum, ΔK is a spectrum width, $\delta K \propto \Delta K / N$ is a spectral width of a single mode. The expression (20) corresponds to an equality $2N \cdot (u^2)_{av} = D$. In the presence of induced interference the amplitude of modulation is given by

$$\frac{N}{\Delta K}\int\sqrt{\frac{D}{N}}\exp(-\frac{|K-K_{N/2}|}{\Delta K})\cdot Sin(K\cdot x)dK=\sqrt{D\cdot\Delta K}\frac{Sin(K_{N/2}\cdot x)}{(\Delta K)^2\cdot x^2+1}. \qquad (21)$$

For modulation instability even under the small threshold crossing, the spectrum width ΔK is insignificantly smaller then the wavenumber of a rapidly growing modulation $K_{N/2}$. The most intensive splash is observed on a spatial interval $\Delta x\propto 1/\Delta K\propto 1/K_{N/2}$, which is visibly less, then an average length of modulation $2\pi/K_{N/2}$. The amplitude of a modulation splash is proportional to $\sqrt{D\cdot N}\propto\sqrt{\frac{D\cdot\Delta K}{\delta K}}$, i.e. it is proportional to the square root of the ratio of spectrum width to the spectral width of a single mode. The value of imperfection level *D*, the value of spectrum width ΔK (or the number of modes *N*) and the value of the amplitude of a modulation splash have the greater, if the level of energy absorption decreases.

During the instability evolution the spectrum width ΔK and the amplitude of a forced interference modulation splash (21) decrease.

Let discuss now the behavior of modulated wave in the neighborhood of the modulation splash. Spatial interval, where the amplitude of modulation is large, is $\frac{1}{\Delta K}\propto\frac{1}{K_{N/2}}\propto\frac{10}{k_0}$ if $k_0\propto 10K_{N/2}$, i.e. that interval is wider then a wave-length of the main mode. In the domain of modulation splash there are a few wave-length of the main mode, if $k_0\propto 20K_{N/2}$. When the system comes to continuous spectrum of instability ($\delta K\to 0, N\to\infty$) a forced interference modulation splash will be infrequent one, but with the significant amplitude.

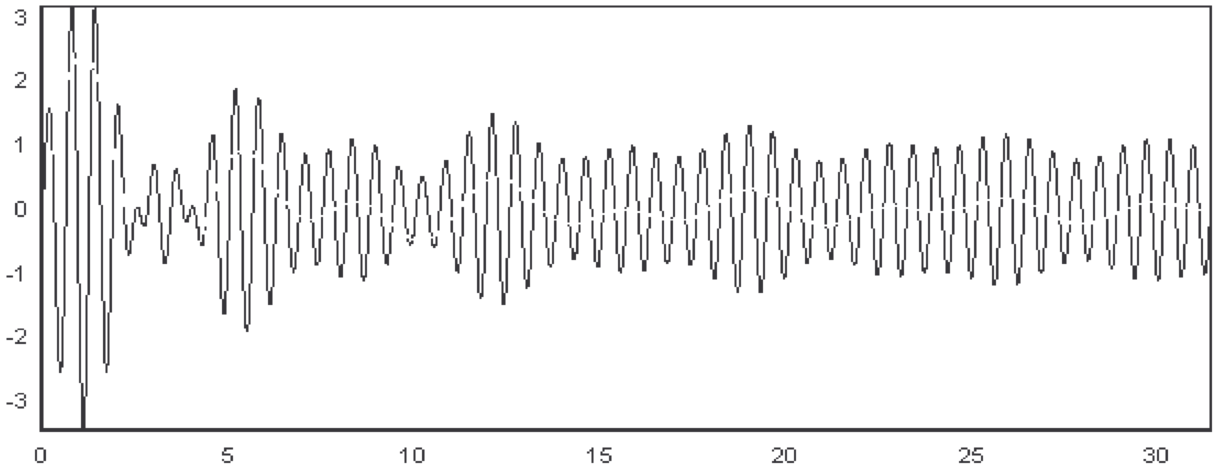

Fig. 2. The wave amplitude behavior in the neighborhood of modulation splash under following conditions: D = 0.7, a variation interval of *x* is equal to 31.4, a wave vector of the main mode $k_0=10,$ a wave vector of the central mode of modulation spectrum $K_{N/2}=1$, a spectrum width of modulation ΔK = 0,8. The white line in the picture corresponds to the envelope of modulation spectrum or, in other words, to the second item in (17).

Thus the phenomena of forced interference of instability spectrum modes causes an appearance of anomalous splash of the fine structure of perturbation amplitude at early stage of nonlinear rate of instability. Frequency of splash appearance is determined by difference of phase velocity of modes. These modes form the wave modulation. The amplitude of splash depends on the number of modes or on phase spectral concentration in the instability spectrum.

4. Formation of line spectrum of developed instability

Modes with the wave numbers $k_n=k_0+K_{N/2}$ and $k_{-n}=k_0-K_{N/2}$ offers the largest linear increment. On instability development the pump level decrease, effective increments of the rest of modes decrease and change into decrements. All that processes follow from the equations (5)-(10). Thus, the mode competition due to mechanism of "pump depletion" results in the bandwidth reduction of developed instability.

From the equations (5)-(10) one finds that an amplitude of the main mode changes from the initial value $u_0(t=0)=1$ to $\sqrt{\delta}$. At the same time the modes of unstable spectrum at first increase, then after the sign reversal of second term of equation (8) (see also (15)) decrease their's amplitudes. Finally only two modes remain, where the wavenumbers are near $k_0 + K_{n*}$, $k_0 - K_{n*}$ (where $K_{n*} \approx \sqrt{\delta}$) and amplitudes are equal to $u_{n*} \propto \sqrt{\delta(1-\delta)/2}$.

Hence the line spectrum composed of the main mode and two lateral satellites are formed. The ratio $D = 2u_{n*}^2 / u_0^2 \approx 1-\delta$ determines a modulation rate or, in other words, a level of imperfection with the spatial scale $2\pi / K_{n*}$. The integrated intensity during an instability evolution doesn't change. It is important that bandwidth reduction of primary instabilities guarantees a fulfillment of the severe requirement for the instability cascade formation, which will be discussed later in this article. Since the bandwidth reduction of the finite amplitude waves always reduces the modulation instability thresholds [18,19].

In nonlinear systems the phase locking causes the regulation and stabilization of a phase position in well-defined reference frame (at the expense of attractors appearance and statical stabilization). In case of the quasilinear operation the phase locking means somewhat different. This is so indeed, only relative phase velocities are given by the pump, which determines in that way the phase dynamics of unstable modes (that is well-ordered dynamics, without statical stabilization).

5. The cascade of instabilities and a self-similar structure formation

The modulation instability of monochromatic finite amplitude wave discussed above forms a new state, i.e. a modulated wave, which in it's turn is unstable [17]. Let's show, that as a result of the secondary modulation instability an even greater large-scale modulation of the earlier modulated wave is formed. For description of the modulation instability the initial state of system is determined by the line spectrum formed by primary modulation instability. This line spectrum is composed of the main mode and two lateral satellites. As a result of secondary instability in neighbourhood of the modes $k_n = k_+^* = k_0 + K_{n*}$ $k_{-n} = k_-^* = k_0 - K_{n*}$ the spectra of perturbation are formed. The modes of that spectra have the wavenumbers, which areequal to $(k_\pm^* \pm \kappa_S)$. The modes with the wavenumbers $(k_+^* + \kappa_S)$ and $(k_-^* - \kappa_S)$ have the same value of amplitude v_s, but not the same phases. The evolution of secondary instability is similar to the primary instability development described above. The pump involves not only the main mode, but also two modes, so-called lateral satellites, survived as a result of primary instability. It is possible to form a fine structure of the large-scale modulation on the quasilinear stage of instability and the line spectrum of mature structures on the saturation stage of instability. The equations of perturbation amplitude with $k = k_0$, $k = k_\pm^*, (k_\pm^* \pm \kappa_s)$ and integrated phases in that case take the form

$$u_0 = -g\{-\delta - 2u_{n*}^2 \sin \Phi_{n*} - 2\sum_{S>0}^{M} v_S^2 Sin\, \Phi'_S\}^{-1}, \tag{22}$$

$$\frac{d\varphi_0}{dt} = +k_0{}^2 - u_0{}^2 - 4u_{n*}^2 - 4\sum_{S>0}^{M} v_S^2 - 2u_{n*}^2 Cos\Phi_{n*} - 2\sum_{S>0}^{M} v_S{}^2 Cos\Phi'_S, \tag{23}$$

$$\frac{du_{n*}}{dt} = u_{n*}\{-\delta + u_0^2 Sin\, \Phi_{n*} - 2\sum_{S>0}^{M} v_S^2 Sin\, \Psi'_S\}, \tag{24}$$

$$\frac{d\varphi_{\pm n*}}{dt} = k_{\pm n*}^2 - 2(u_0^2 + \frac{3}{2}u_{n*}^2 + 2\sum_{S>0}^{M} v_S^2) - u_0^2 Cos\,\Phi_{n*} - 2\sum_{S>0}^{M} v_S^2 Cos\,\Psi'_S, \tag{25}$$

$$\frac{dv_S}{dt} = v_s\{-\delta + u_0^2 Sin\, \Phi'_S + 2u_{n*}^2 Sin\, \Psi'_S\}, \tag{26}$$

$$\frac{d\varphi_{\pm S}}{dt} = k_{\pm S}^2 - 2(u_0^2 + \frac{3}{2}v_S^2 + 2u_{n*}^2) - u_0^2 Cos\,\Phi'_S - 2u_{n*}^2 Cos\,\Psi'_S, \tag{27}$$

where $\Phi_{n*} = 2\varphi_0 - \varphi_{n*} - \varphi_{-n*}$, $\Phi'_s = 2\varphi_0 - \varphi_S - \varphi_{-S}$, $\Psi'_S = \varphi_{n*} + \varphi_{-n*} - \varphi_S - \varphi_{-S}$, it is obvious that $\Psi'_S = \Phi'_S - \Phi_{n*}$, v_S and φ_S - amplitude and phase s-th mode of secondary instability. The equations for phases Φ_{n*} and Ψ'_S one may obtain, using the previous equations, the form

$$\begin{aligned} \frac{d\Phi_{N/2}}{dt} &= \Delta_{n*} + 2(u_0^2 - u_{n*}^2) + 2(u_0^2 - 2u_{n*}^2)Cos\,\Phi_{n*} + \\ &+ 4\sum_{S>0}^{M} v_S^2 Cos\,\Psi'_S - 4\sum_{S>0}^{M} v_S^2 Cos\,\Phi'_S \end{aligned} \tag{28}$$

$$\frac{d\Psi_S^{'}}{dt} = \Delta_{n^*,S} + 2(u_{n^*}^2 + 3v_S^2 - 4\sum_{S'>0}^{M} v_{S'}^2) - 2u_0^2 Cos\,\Phi_{n^*} +$$
$$+ 2u_0^2 Cos\,\Phi_S^{'} - 4\sum_{S'>0}^{M} v_{S'}^2 Cos\,\Psi_{S'}^{'} + 4u_{n^*}^2 Cos\,\Psi_S^{'} \quad (29)$$

where $\Delta_{n^*} = 2k_0^2 - (k_0 + K_{n^*})^2 - (k_0 - K_{n^*})^2$, $\Delta_S = 2k_0^2 - (k_0 + K_{n^*} + \kappa_S)^2 - (k_0 - K_{n^*} - \kappa_S)^2$
$\Delta_{n^*,S} = (k_0 + K_{n^*})^2 + (k_0 - K_{n^*})^2 - (k_0 + K_{n^*} + \kappa_S)^2 - (k_0 - K_{n^*} - \kappa_S)^2$, at that $\Delta_{n^*,S} = \Delta_S - \Delta_{n^*}$.

Linear theory. From analysis of dynamics of primary modulation instability it is clear, that $\Phi_{n^*} \approx \pi/2$, and the equations (26) and (29) transform as follows

$$\frac{dv_S}{dt} = v_s\{-\delta + \sqrt{u_0^4 + 4u_{n^*}^4}\,Sin\,(\Psi_S^{'} + \alpha)\}\ , \quad (30)$$

$$\frac{d\Psi_S^{'}}{dt} = \Delta_{n^*,S} + 2(u_{n^*}^2) + 2\sqrt{u_0^4 + 4u_{n^*}^4}\,Cos\,(\Psi_S^{'} + \alpha)\,, \quad (31)$$

where α is a stationary phase. Let us assume that the change of every *M*-chanal phase $\Psi_s^{'}$ on the linear stage of instability is not essential. Then from the expression

$$Cos(\Psi_S^{'} + \alpha) = -[\Delta_{n^*,S} + 2(u_{n^*}^2)]/2\sqrt{u_0^4 + 4u_{n^*}^4}\ , \quad (32)$$

one can find a maximum increment of the secondary modulation instability

$$(\mathrm{Im}\,\omega)_{MAX}^{(2)} = \sqrt{\delta^2 + (1-\delta)^2} - \delta\,, \quad (33)$$

which is realized if $-\Delta_{n^*,M/2} \approx 2K_{n^*}\cdot\kappa_{M/2} = 2(u_{n^*}^2) \approx (1-\delta)$, i.e. $\kappa_{M/2} \approx (1-\delta)/2$

Thus the ratio of maximum increment of the secondary and the primary modulation instability is

$$[\sqrt{\delta^2 + (1-\delta)^2} - \delta]/(1-\delta) \propto (1-\delta)/2 \text{ при } (1-\delta) << 1\ . \quad (34)$$

If $K_{n^*} = \sqrt{\delta} \propto 1$, the ratio of a scale of the secondary and primary modulation is equal to $\kappa_* / K_{n^*} \propto (1-\delta)/2$, where $\kappa_* \approx \kappa_{M/2}$.

Evolution of instability cascade. It is easy to show from the equations (22)-(29), that as the result of evolution of the secondary instability the bandwidth reduction occurs. In fact, there are modes with the wave numbers $(k_\pm^* \pm \kappa_*)$ in the spectrum of developed secondary instability. The developed secondary instability in it's turn leads to growth of the next cascade of instability. The subsequent cascades of instability form even larger structures in the large-scale region, i.e. modulations. Let us note that the intensity of modulated wave at every cascade of instability practically does not change. Here just the energy redistribution among the modes takes place. The modes, which appear on the previous cascade give their own energy to the partially modes, which have win the hard competition during the subsequent instabilities.

Acknowledgment

Author is indebted to Gennady Fraiman, Vladimyr Pavlenko and Anatoly Zagorodny for useful discussions.

1.Lighhill M. J. *J. Inst. Math. Appl.*V.1, N.2. 269-306 ,(1965).
2. Zakharov V.E. *J.Teor. Prikl. Fiz.* 51, 668-678 (1966); *Sov. Phys. JETP* 24, 455-470, (1967).
3.Benjamin T.B. and Feir J.E., *J. Fluid Mech*. 27, 417-425 (1967).
4.Feir J.E., *Proc. R. Soc. Lond.* A.299, 54,(1967).
5.Rasmussen J.J. Rupdal K. *Physica Scripta*.- 1988. V.33, N.2.- p. 481 – 497.
6.Zakharov V.E. *J. Appl. Mech. Tech. Phys.* 9, 190 (168).
7.Zakharov V.E. and Shabat A.B., *Sov. Phys. JETP* 34, 62 (1972).
8. Dysthe K.B., *Proc. Roy.* A369, 105, (1979).
9.Trulsen K. and Dysthe K.B., *Wave motion* 24, 281 (1996).
10. Vorobyov V.M., Kuklin V.M., *Pi'sma Zh. Techn. Fiz.*V.13, N.22, 1354 -1357, (1987).
11.Kuklin V.M. and Panchenko I.P. *Plasma Physics Reports*, V.20, N.9, 813-823 (1994).
12. Kuklin V.M. *Ukr. J. Phys. Reviews,* V.1, N.1, 49-81, (2004).
13. Horsthemke W., Lefever R. *Noise-Induced Transitions.* Springer-Verlag, Berlin, Heidelberg, New-York, Tokio, 1984, 380 p.
14. Kirichok A.V. and Kuklin V.M. *Physics and Chemistry of the Earth.* Part A № 6, 533-538, (1999).

15.Kuklin V.M. *Electromagnetic Phenomena,* V5, №2 (15), 67-78, (2005).
16. Kuklin V.M.*Khariton Scientific Lecture*. *Abstracts*, Sarov, 126-127, March 21-24, (2006).
17. Kuklin V.M., Kirichok A.V. Kuklina O.V. *Vopr. Atom. Nauk,Tech.* №1, 222-224, (2000).
18. Valeo E.J. and Olerman C.R. Phys. Rev. Lett. V.30, N21, 1035-1038, (1973).
19.Thomson J.J. and Karush J.I. *Physics of Fluids,* V.17, N8, 1608-1613, (1974).
20. Newell A.C. Solitons and Condensed Matter Physics (Bishop and Schneider, Eds). – Berlin: Springer, 1978.- p.520.